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Two classes of pseudo-triangular norms and fuzzy implications[☆]

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ABSTRACT

Two kinds of extensions of triangular norms (t-norms) are proposed, and the relations between these extensions and fuzzy implications are discussed in this paper. First, two classes of pseudo-t-norms (ps-t-norms), called type-A and type-B ps-t-norms, and their respective residual operators are defined. Then, we prove that these residual operators are fuzzy implications and satisfy the left neutral property. For these two classes of pseudo-t-norms, we give a series of equivalent conditions of left-continuity with respect to their first or second variable. By combining the axioms of the two classes of pseudo-t-norms, we simply get the definition of the triangular seminorms. Furthermore, we define two classes of induced operators from fuzzy implications and give the conditions under which they are type-A ps-t-norms, type-B ps-t-norms and t-seminorms. For a fuzzy implication, a series of equivalent conditions of right-continuity with respect to its second variable are established. Finally, another method inducing type-A ps-t-norms, type-B ps-t-norms and t-seminorms by implications is proposed.

1. Introduction

In fuzzy logics, the set of truth values of fuzzy propositions is modeled by the unit interval $[0, 1]$, and the truth function for a conjunction connective is usually taken as a triangular norm (t-norm for short) on $[0, 1]$ which is monotone, associative, commutative and has neutral element 1 (see [1]). But the t-norms are inadequate to deal with natural interpretations of linguistic words since the axioms of t-norms are quite strong. For instance, when we say "she is very beautiful but stupid", this is not equivalent to "she is very beautiful and stupid". It is in fact "she is very beautiful & stupid" in such a way that & is not a commutative connective but "and" is the common commutative conjunction (see [2]). In order to interpret the non-commutative conjunctions, Flondor et al. [3] introduced non-commutative t-norms by throwing away the axiom of commutativity of t-norms and used them to construct pseudo-BL-algebras and weak pseudo-BL-algebras (i.e., pseudo-MTL-algebras [4]). As regards another axiom of t-norms, i.e. associativity, as stressed in [5,6], for example, "if one works with binary conjunctions and there is no need to extend them for three or more arguments, as happens e.g. in the inference pattern called generalized modus ponens, associativity of the conjunction is an unnecessarily restrictive condition". By removing the commutativity and associativity axioms of t-norms, Fodor [7,8] proposed weak t-norms on $[0, 1]$ and discussed the relations between weak t-norms and implications. Noticing that the QL-implications on $[0, 1]$ cannot be induced by weak t-norms on $[0, 1]$, Wang and Yu [9] generalized the notion of weak t-norms and introduced pseudo-t-norms on a complete Brouwerian lattice L. Further, the relation between the pseudo-t-norms and implications on L was discussed in [9].

Fuzzy implications play an important role in approximate reasoning, fuzzy control and many other theoretical and application fields (see, e.g., [10–14]). In the literature, there are several different definitions regarding fuzzy implications. The one used by Wang and Yu [9] is as follows.

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Definition 1.1. A function $I : [0, 1]^2 \to [0, 1]$ is called a fuzzy implication if it is increasing in its second variable and satisfies $I(1, x) = x$ and $I(0, x) = 1$ for all $x \in [0, 1]$.

In this definition, the neutrality property (NP), i.e., $I(1, x) = x$ for all $x \in [0, 1]$, is treated as an axiom. In practical applications, however, many fuzzy implications employed do not necessarily satisfy (NP). For instance, the following basic fuzzy implications do not satisfy (NP) [14,15]: for any $x, y \in [0, 1]$,

$$I_{MN}(x, y) = \begin{cases} \min(1-x, y), & \text{if } \max(1-x, y) \le 0.5, \\ \max(1-x, y), & \text{otherwise}, \end{cases}$$

$$I_{BZ}(x, y) = \min(\max(0.5, \min(1-x+y, 1)), 2-2x+2y),$$

$$I_{RS}(x, y) = \begin{cases} 1, & \text{if } x \le y, \\ 0, & \text{if } x > y. \end{cases}$$

In addition, in real applications, we usually need a fuzzy implication I to be decreasing in its first variable. This left antitonicity of I gives the fuzzy implication its unique flavor. It captures the idea that a decrease in the truth value of the antecedent increases its efficacy to state more about the truth value of its consequent [14]. So, at present, the extensively used definition of fuzzy implications in research and applications is the following form, proposed by Fodor and Roubens [10].

Definition 1.2. A function $I : [0, 1]^2 \to [0, 1]$ is called a fuzzy implication if it is decreasing in its first variable, increasing in its second one, and fulfils the following implication truth table: $I(0, 0) = I(0, 1) = I(1, 1) = 1, I(1, 0) = 0$.

The function $N : [0, 1] \to [0, 1]$ defined by $N(x) = I(x, 0)$ for all $x \in [0, 1]$ is called the natural negation induced by the fuzzy implication I.

It follows from the definition that $I(0, x) = 1$ and $I(x, 1) = 1$ for all $x \in [0, 1]$, whereas the symmetrical values $I(x, 0)$ and $I(1, x)$ are not derived from the definition.

In our study, we notice that Wang and Yu's pseudo-t-norms do not always generate fuzzy implications in the sense of Definition 1.2 by means of the ordinary residual techniques. So we redefine the pseudo-t-norms in the present paper and propose two kinds of pseudo-t-norms, called type-A and type-B ps-t-norms, respectively. Combining the axioms of types A and B ps-t-norms we simply get the definition of t-seminorms [16] (also called semi-copula [17,18]). We also discuss the residual implications of the new pseudo-t-norms and propose the methods to induce the new pseudo-t-norms and t-seminorms by fuzzy implications.

The rest of this paper is organized as follows. Section 2 defines the two classes of pseudo-t-norms called type-A and type-B ps-t-norms and their respective residual operators, and discuss some of their properties. In Section 3, we define two kinds of induced operators from fuzzy implications and discuss the conditions such that they are type-A ps-t-norms, type-B ps-t-norms or t-seminorms. Some properties related to fuzzy implications and the induced operators are also investigated. We also propose another inducing method for the two classes of pseudo-t-norms and t-seminorms. Section 4 presents our conclusions.

2. Pseudo-t-norms and their residual implications

In the following, unless otherwise stated, we always assume that fuzzy implications are in the sense of Definition 1.2, and denote the set of all fuzzy implications by $\mathcal{F}\mathcal{I}$. The expression "pseudo-t-norm" is written as "ps-t-norm" for short.

Additional properties of fuzzy implications have been postulated in many works (see [10,11,14,19–21]). We now list two of them for our usage.

Definition 2.1. A fuzzy implication I is said to have

(NP) the left neutrality property if $I(1, y) = y$ for all $y \in [0, 1]$;
(OP) the order property if $I(x, y) = 1 \Leftrightarrow x \le y$ for all $x, y \in [0, 1]$.

2.1. Type-A ps-t-norms and their residual implications

Definition 2.2. A function $T : [0, 1]^2 \to [0, 1]$ is called a type-A ps-t-norm if it satisfies the following.

(i) $T(x, 1) = x$, $T(x, 0) = 0$ for all $x \in [0, 1]$.
(ii) $y \le z$ implies that $T(x, y) \le T(x, z)$ for all $x, y, z \in [0, 1]$.

From this definition we know that $T(x, y) \le x$ for all $x, y \in [0, 1]$ since $T(x, y) \le T(x, 1) = x$. So we have $T(0, y) = 0$ for all $y \in [0, 1]$. But the property $T(1, x) = x$ for all $x \in [0, 1]$ cannot be derived from the definition. For instance, the function $T : [0, 1]^2 \to [0, 1]$ defined by

$$T(x, y) = \begin{cases} x, & y = 1 \\ 0, & y = 0 \\ 1, & \text{otherwise} \end{cases}, \quad x, y \in [0, 1]$$

is a type-A ps-t-norm, but $T(1, y) \ne y$ for all $y \in (0, 1)$.

By the ordinary residual technique, we now define the residual operators for the type-A ps-t-norms.

Definition 2.3. Let T be a type-A ps-t-norm on $[0, 1]$. The following $I_{1T} : [0, 1]^2 \to [0, 1]$ is called the residual operator of T:

$$I_{1T}(x, y) = \sup\{t \in [0, 1] \mid T(t, x) \le y\}, \quad x, y \in [0, 1]. \tag{2.1}$$

Similar to the case of t-norms [1,22], one can associate a fuzzy negation to any type-A ps-t-norm.

Definition 2.4. Let T be a type-A ps-t-norm on $[0, 1]$. A function $N_{1T} : [0, 1] \to [0, 1]$ defined by

$$N_{1T}(x) = \sup\{t \in [0, 1] \mid T(t, x) = 0\}, \quad x \in [0, 1] \tag{2.2}$$

is called the natural negation of T.

Obviously, the appropriate set in (2.2) is nonempty since 0 is in it.

Remark 2.1. (i) It is easy to prove that N_{1T} is a fuzzy negation, i.e., it is decreasing and satisfies $N_{1T}(1) = 0$ and $N_{1T}(0) = 1$.
(ii) It follows from (2.2) that $x \le N_{1T}(y)$ if $T(x, y) = 0$ for some $x, y \in [0, 1]$. Conversely, assume that type-A ps-t-norm T is increasing in its first variable; if $x < N_{1T}(y)$, then $T(x, y) = 0$. In fact, there exists $x_0 \in [0, 1]$ satisfying $T(x_0, y) = 0$ such that $x < x_0$ if $x < N_{1T}(y)$. Thus, $T(x, y) \le T(x_0, y) = 0$.

Theorem 2.1. *Let T be a type-A ps-t-norm on $[0, 1]$; then $I_{1T} \in \mathcal{FI}$. Moreover, I_{1T} satisfies (NP), and the natural negation of I_{1T} is also the natural negation of T.*

Proof. First, we show that $I_{1T} \in \mathcal{FI}$. The increasingness of T in its second variable implies that I_{1T} is decreasing in its first variable. The fact that I_{1T} is increasing in its second variable is obvious from (2.1). Moreover, it is easy to check that $I_{1T}(0, 0) = I_{1T}(0, 1) = I_{1T}(1, 1) = 1$ and $I_{1T}(1, 0) = 0$. Therefore, $I_{1T} \in \mathcal{FI}$. Further, I_{1T} satisfies (NP) since $I_{1T}(1, y) = \sup\{t \in [0, 1] | T(t, 1) \le y\} = \sup\{t \in [0, 1] | t \le y\} = y$ for any $y \in [0, 1]$. The fact that $N_{I_{1T}} = N_{1T}$ can be seen from (2.2). □

Theorem 2.2. *Suppose that type-A ps-t-norm T on $[0, 1]$ is increasing in its first variable; then the following statements are equivalent:*

(i) *T is left-continuous in its first variable;*
(ii) *$T(x, y) \le z$ iff $x \le I_{1T}(y, z)$ for all $x, y, z \in [0, 1]$;*
(iii) *$I_{1T}(x, y) = \max\{t \in [0, 1] \mid T(t, x) \le y\}$ for any $x, y \in [0, 1]$;*
(iv) *$T(I_{1T}(x, y), x) \le y$ for any $x, y \in [0, 1]$.*

Proof. (i) ⇒ (ii). For any $x, y, z \in [0, 1]$, $T(x, y) \le z$ implies that $x \in \{t \in [0, 1] | T(t, y) \le z\}$, and hence $x \le I_{1T}(y, z)$. Conversely, assume that $x \le I_{1T}(y, z)$ for some $x, y, z \in [0, 1]$. The facts that T is increasing and left-continuous in its first variable imply that T is infinitely sup-distributive in its first variable. So we have $T(x, y) \le T(I_{1T}(y, z), y) = T(\sup\{t \in [0, 1] | T(t, y) \le z\}, y) = \sup\{T(t, y) | t \in [0, 1], T(t, y) \le z\} \le z$.

(ii) ⇒ (iii). For any $x, y \in [0, 1]$, the inequality $I_{1T}(x, y) \le I_{1T}(x, y)$ implies that $T(I_{1T}(x, y), x) \le y$, which means that the supremum in (2.1) is the maximum.

(iii) ⇒ (i). Since T is increasing in its first variable, it is enough to show that T is infinitely sup-distributive in its first variable, i.e., $T(\sup_{x\in X} x, y) = \sup_{x\in X} T(x, y)$, where X is any subset of $[0, 1]$. First, the previous equality holds for $X = \emptyset$ since $\sup \emptyset = 0$ and $T(0, y) = 0$. For the case of $X \ne \emptyset$, it follows from the monotonicity of T that $T(\sup_{x\in X} x, y) \ge \sup_{x\in X} T(x, y)$. We now write $z = \sup_{x\in X} T(x, y)$. Then $T(x, y) \le z$ for all $x \in X$, and hence $x \in \{t \in [0, 1] | T(t, y) \le z\}$ for any $x \in X$, which means that $x \le I_{1T}(y, z)$ for any $x \in X$. Thus $\sup_{x\in X} x \le I_{1T}(y, z)$. From the monotonicity of T and (iii) we get $T(\sup_{x\in X} x, y) \le T(I_{1T}(y, z), y) \le z = \sup_{x\in X} T(x, y)$. Summarizing the above, we have proved that $T(\sup_{x\in X} x, y) = \sup_{x\in X} T(x, y)$.

We now prove the equivalence between (ii) and (iv).

(ii) ⇒ (iv). It has been proved in (ii) ⇒ (iii).

(iv) ⇒ (ii). If $T(x, y) \le z$, then $x \in \{t \in [0, 1] | T(t, y) \le z\}$. So we get $x \le I_{1T}(y, z)$. Conversely, if $x \le I_{1T}(y, z)$, then it follows from the monotonicity of T and (iv) that $T(x, y) \le T(I_{1T}(y, z), y) \le z$. □

Theorem 2.3. *If type-A ps-t-norm T on $[0, 1]$ is increasing and left-continuous in its first variable, then $T(I_{1T}(y, T(x, y)), y) = T(x, y)$ and $I_{1T}(x, T(I_{1T}(x, y), x)) \le I_{1T}(x, y)$ hold for any $x, y \in [0, 1]$.*

Proof. Since T is increasing and left-continuous in its first variable, it is infinitely sup-distributive w.r.t. this variable. For any $x, y \in [0, 1]$, we get

$$\begin{aligned} T(I_{1T}(y, T(x, y)), y) &= T(\sup\{t \in [0, 1] | T(t, y) \le T(x, y)\}, y) \\ &= \sup\{T(t, y) | t \in [0, 1], T(t, y) \le T(x, y)\} = T(x, y); \end{aligned}$$

$$\begin{aligned} I_{1T}(x, T(I_{1T}(x, y), x)) &= I_{1T}(x, T(\sup\{t \in [0, 1] | T(t, x) \le y\}, x)) \\ &= I_{1T}(x, \sup\{T(t, x) | t \in [0, 1], T(t, x) \le y\}) \le I_{1T}(x, y). \quad \square \end{aligned}$$

2.2. Type-B ps-t-norms and their residual implications

In 2002, Wang and Yu gave the following definition of pseudo-t-norms.

Definition 2.5 (*Wang and Yu [9]*). A function $T : [0,1]^2 \to [0,1]$ is called a pseudo-t-norm if it satisfies the following conditions.

(i) $T(1,y) = y$, $T(0,y) = 0$ for any $y \in [0,1]$.
(ii) $y \le z$ implies that $T(x,y) \le T(x,z)$ for all $x, y, z \in [0,1]$.

Wang and Yu [9] also defined the residual operator $I_T : [0,1]^2 \to [0,1]$ as follows:

$$I_T(x,y) = \sup\{t \in [0,1] \mid T(x,t) \le y\}, \quad x, y \in [0,1]. \tag{2.3}$$

Remark 2.2. For Wang and Yu's pseudo-t-norm T, I_T defined by (2.3) is not necessarily a fuzzy implication in the sense of Definition 1.2. For instance, the function $T : [0,1]^2 \to [0,1]$ defined by

$$T(x,y) = \begin{cases} y, & x = 1 \\ 0, & x = 0 \\ 1, & \text{otherwise} \end{cases}, \quad x, y \in [0,1]$$

is a pseudo-t-norm, but its residual operator I_T with the form

$$I_T(x,y) = \begin{cases} y, & x = 1 \\ 1, & x = 0 \text{ or } y = 1 \\ 0, & \text{otherwise} \end{cases}, \quad x, y \in [0,1]$$

is not a fuzzy implication in the sense of Definition 1.2, because it is not decreasing in its first variable. For this reason, we propose a new kind of pseudo-t-norm by revising Wang and Yu's definition, and we call them type-B ps-t-norms.

Definition 2.6. A function $T : [0,1]^2 \to [0,1]$ is called a type-B ps-t-norm if it satisfies the following.

(i) $T(1,y) = y$, $T(0,y) = 0$ for all $y \in [0,1]$.
(ii) $x \le y$ implies that $T(x,z) \le T(y,z)$ for all $x, y, z \in [0,1]$.

The residual operator of a type-B ps-t-norm T is defined by (2.3), i.e.,

$$I_{2T}(x,y) = \sup\{t \in [0,1] \mid T(x,t) \le y\}, \quad x, y \in [0,1].$$

The natural negation of a type-B ps-t-norm T is defined by

$$N_{2T}(x) = \sup\{t \in [0,1] \mid T(x,t) = 0\}, \quad x \in [0,1]. \tag{2.4}$$

Obviously, if a type-B ps-t-norm T is commutative, then it is also a type-A ps-t-norm. For any type-B ps-t-norm T, the function T' defined by $T'(x,y) = T(y,x)(x, y \in [0,1])$ is a type-A ps-t-norm, and we have $I_{2T} = I_{1T'}$ and $N_{2T} = N_{1T'}$. So, from the previous subsection, we have the following results for type-B ps-t-norms.

Theorem 2.4. *Let T be a type-B ps-t-norm on $[0,1]$; then I_{2T} is a fuzzy implication and it satisfies (NP). Moreover, $N_{I_{2T}} = N_{2T}$.*

Remark 2.3. Theorems 2.1 and 2.4 told us that I_{1T} and I_{2T} are also fuzzy implications in the sense of Definition 1.1.

Theorem 2.5. *Assume that type-B ps-t-norm T on $[0,1]$ is increasing in its second variable; then the following statements are equivalent:*

(i) *T is left-continuous in its second variable;*
(ii) *$T(x,y) \le z$ if and only if $y \le I_{2T}(x,z)$ for any $x, y, z \in [0,1]$;*
(iii) *$I_{2T}(x,y) = \max\{t \in [0,1] \mid T(x,t) \le y\}$ for any $x, y \in [0,1]$;*
(iv) *$T(x, I_{2T}(x,y)) \le y$ for all $x, y \in [0,1]$.*

The equivalence between (i), (ii) and (iii) is the same as the result for pseudo-t-norms given by Wang and Yu [9]. It is clear that a left-continuous and increasing (in its second variable) type-B ps-t-norm T and its residual implication I_{2T} satisfy the GMP in the form $T(x, I_{2T}(x,y)) \le y$ for all $x, y \in [0,1]$.

Theorem 2.6. *If type-B ps-t-norm T on $[0,1]$ is increasing and left-continuous in its second variable, then $T(x, I_{2T}(x, T(x,y))) = T(x,y)$, $I_{2T}(x, T(x, I_{2T}(x,y))) = I_{2T}(x,y)$ for all $x, y \in [0,1]$.*

By combining the axioms of type-A and type-B ps-t-norms, we simply obtain the definition of t-seminorms [16] (also called semi-copula [17,18]).

Definition 2.7. A function $T : [0,1]^2 \to [0,1]$ is called a triangular seminorm (briefly t-seminorm) if it satisfies the following conditions.

(i) $T(1,x) = T(x,1) = x$ for all $x \in [0,1]$.
(ii) T is increasing in each variable.

Remark 2.4. (i) It is clear from the above definition that $T(x,0) = T(0,x) = 0$ and $T_D(x,y) \le T(x,y) \le T_M(x,y)$ for any $x, y \in [0,1]$, where $T_D(x,y) = \min(x,y)$ if $\max(x,y) = 1$, $T_D(x,y) = 0$ otherwise, and $T_M(x,y) = \min(x,y)$ [22].
(ii) The above definition can also be seen to be obtained by adding conditions $T(x,1) = x(\forall x \in [0,1])$ and that T is increasing in its first variable to Wang and Yu's pseudo-t-norms, or by strengthening the conditions of Fodor's weak t-norms via replacing the axiom $T(x,1) \le x$ by $T(x,1) = x$ for all $x \in [0,1]$ (see [7,8]). So, any t-seminorm must be a weak t-norm and hence a pseudo-t-norm. A t-seminorm is a t-norm only when it is commutative and associative.

For a left-continuous t-seminorm T on $[0,1]$, I_{1T} and I_{2T} have a series of nice properties (see Theorem 3.2 in [18]).

3. Types A and B ps-t-norms and t-seminorms induced by fuzzy implications

Definition 3.1. Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication. We define the induced operators T_{1I} and T_{2I} from I as follows, for any $x, y \in [0,1]$:

$$T_{1I}(x,y) = \inf\{t \in [0,1] \mid x \le I(y,t)\} \tag{3.1}$$

$$T_{2I}(x,y) = \inf\{t \in [0,1] \mid y \le I(x,t)\}. \tag{3.2}$$

Obviously, $T_{1I} = T_{2I}$ holds if I satisfies $x \le I(y,z)$ iff $y \le I(x,z)$ for any $x, y, z \in [0,1]$.

Remark 3.1. (i) T_{1I} and T_{2I} defined by (3.1) and (3.2) are two well-defined operators, i.e., the appropriate sets in (3.1) and (3.2) are non-empty since $I(x,1) = 1$ for any $x \in [0,1]$.
(ii) It is worthwhile mentioning that T_{1I} and T_{2I} defined by the above are not necessarily type-A or type-B ps-t-norms. For instance, if we take the fuzzy implication I,

$$I(x,y) = \begin{cases} 0, & \text{if } x > 0 \text{ and } y = 0 \\ 1, & \text{otherwise,} \end{cases}$$

then, for any $x > 0$, by (3.1) and (3.2), we obtain that $T_{1I}(x,1) = 0 \ne x$ and $T_{2I}(1,x) = 0 \ne x$. These facts mean that I_{1T} is not a type-A ps-t-norm and I_{2T} is also not a type-B ps-t-norm.

The following theorem gives the conditions under which T_{1I} and T_{2I} are respectively type-A and type-B ps-t-norms.

Theorem 3.1. *Let I be a fuzzy implication on $[0,1]$ satisfying (NP); then T_{1I} and T_{2I} defined by (3.1) and (3.2) are respectively type-A and type-B ps-t-norms.*

Proof. We only prove that T_{1I} is a type-A ps-t-norm since the other proof is similar. For any $x \in [0,1]$, $T_{1I}(x,1) = \inf\{t \in [0,1] \mid x \le I(1,t)\} = \inf\{t \in [0,1] \mid x \le t\} = x$, $T_{1I}(x,0) = \inf\{t \in [0,1] \mid x \le I(0,t)\} = 0$, since $I(0,t) = 1$ for all $t \in [0,1]$. For any $y, z \in [0,1]$ and $y \le z$, we have $I(z,t) \le I(y,t)$ for any $t \in [0,1]$, and hence $\{t \in [0,1] | x \le I(z,t)\} \subseteq \{t \in [0,1] | x \le I(y,t)\}$. So we get $T_{1I}(x,y) \le T_{1I}(x,z)$ if $y \le z$. Therefore T_{1I} is a type-A ps-t-norm. □

Remark 3.2. T_{1I} and T_{2I} defined by the above are not necessarily t-seminorms. For instance, if we take $I(x,y) = 1 - x + xy$ for all $x, y \in [0,1]$ (Reichenbach implication), then, for any $x > 0$, by (3.1) and (3.2), we obtain that $T_{1I}(1,x) = 1 \ne x$ and $T_{2I}(x,1) = 1 \ne x$. These facts mean that I_{1T} and I_{2T} are not t-seminorms.

We now give conditions such that T_{1I} and T_{2I} are t-seminorms.

Theorem 3.2. *Let I be a fuzzy implication on $[0,1]$ satisfying (NP) and (OP); then T_{1I} and T_{2I} defined by (3.1) and (3.2) are t-seminorms.*

Proof. Since implication I satisfies (NP) and (OP), we get by (3.1) that, for any $x \in [0,1]$,

$$T_{1I}(1,x) = \inf\{t \in [0,1] \mid 1 \le I(x,t)\} = \inf\{t \in [0,1] \mid x \le t\} = x,$$
$$T_{1I}(x,1) = \inf\{t \in [0,1] \mid x \le I(1,t)\} = \inf\{t \in [0,1] \mid x \le t\} = x.$$

For any $x_1, x_2, y_1, y_2 \in [0,1]$ and $x_1 \le x_2, y_1 \le y_2$, since $y_1 \le y_2$ implies that $I(y_2,t) \le I(y_1,t)$ for any $t \in [0,1]$, we have $x_1 \le x_2 \le I(y_2,t) \le I(y_1,t)$, i.e., $t \in \{t \in [0,1] \mid x_1 \le I(y_1,t)\}$, if $t \in \{t \in [0,1] \mid x_2 \le I(y_2,t)\}$. This means that $\{t \in [0,1] \mid x_2 \le I(y_2,t)\} \subseteq \{t \in [0,1] \mid x_1 \le I(y_1,t)\}$. So we get $\inf\{t \in [0,1] \mid x_1 \le I(y_1,t)\} \le \inf\{t \in [0,1] \mid x_2 \le I(y_2,t)\}$, i.e., $T_{1I}(x_1,y_1) \le T_{1I}(x_2,y_2)$ when $x_1 \le x_2$ and $y_1 \le y_2$.

Therefore, T_{1I} is a t-seminorm. The proof for T_{2I} is similar to the above. □

Theorem 3.3. *Let I be a fuzzy implication on* $[0, 1]$*; then the following statements are equivalent:*

(i) *I is right-continuous in its second variable;*
(ii) *$T_{1I}(x, y) \le z$ iff $x \le I(y, z)$ for any $x, y, z \in [0, 1]$;*
(iii) *$T_{1I}(x, y) = \min\{t \in [0, 1] \mid x \le I(y, t)\}$ for any $x, y \in [0, 1]$;*
(iv) *$x \le I(y, T_{1I}(x, y))$ for any $x, y \in [0, 1]$.*

Proof. (i) $\Rightarrow$ (ii). For any $x, y, z \in [0, 1]$, $x \le I(y, z)$ implies that $z \in \{t \in [0, 1]|x \le I(y, t)\}$, and hence $T_{1I}(x, y) \le z$ by (3.1). Conversely, assume that $T_{1I}(x, y) \le z$ for some $x, y, z \in [0, 1]$. The fact that I is increasing and right-continuous in its second variable implies that I is infinitely inf-distributive in its second variable. So we have $I(y, z) \ge I(y, T_{1T}(x, y)) = I(y, \inf\{t \in [0, 1]|x \le I(y, t)\}) = \inf\{I(y, t)|t \in [0, 1], x \le I(y, t)\} = x$.

(ii) $\Rightarrow$ (iii). For any $x, y \in [0, 1]$, the inequality $T_{1I}(x, y) \le T_{1I}(x, y)$ implies that $x \le I(y, T_{1I}(x, y))$, which means that the infimum in (3.1) is the minimum.

(iii) $\Rightarrow$ (i). Since I is increasing in its second variable, it is enough to show that I infinitely inf-distributive in its second variable, i.e., $I(x, \inf_{y\in Y} y) = \inf_{y\in Y} I(x, y)$ holds for any $x \in [0, 1]$, where Y is any subset of $[0, 1]$. First, the previous equality holds for $Y = \emptyset$ since $\inf \emptyset = 1$ and $I(x, 1) = 1$. For the case of $Y \ne \emptyset$, it follows from the increasingness of I w.r.t. its second variable that $I(x, \inf_{y\in Y} y) \le \inf_{y\in Y} I(x, y)$. By writing $z = \inf_{y\in Y} I(x, y)$, $z \le I(x, y)$ for all $y \in Y$, and hence $y \in \{t \in [0, 1]|z \le I(x, t)\}$ for all $y \in Y$, which means that $y \ge T_{1I}(z, x)$. Thus, $\inf_{y\in Y} y \ge T_{1I}(z, x)$. By using the increasingness of I in its second variable and (iii) we get $I(x, \inf_{y\in Y} y) \ge I(x, T_{1I}(z, x)) \ge z = \inf_{y\in Y} I(x, y)$. So we have proved that $I(x, \inf_{y\in Y} y) = \inf_{y\in Y} I(x, y)$.

In what follows, we prove the equivalence between (ii) and (iv). First, (ii) $\Rightarrow$ (iv) has been proved in the process of (ii) $\Rightarrow$ (iii).

(iv) $\Rightarrow$ (ii). If $x \le I(y, z)$, then $z \in \{t \in [0, 1]|x \le I(y, t)\}$, which means that $T_{1I}(x, y) \le z$ by (3.1). Conversely, if $T_{1I}(x, y) \le z$, then it follows from the increasingness of I in its second variable and (iv) that $I(y, z) \ge I(y, T_{1I}(x, y)) \ge x$. □

It is easy to verify from the above theorem that a right-continuous fuzzy implication I in its second variable and its induced operator T_{1I} satisfy the GMP in the form $T_{1I}(I(x, y), x) \le y$ for any $x, y \in [0, 1]$.

Similarly, we have the results for the case of T_{2I}.

Theorem 3.4. *Let I be a fuzzy implication on* $[0, 1]$*; then the following statements are equivalent:*

(i) *I is right-continuous in its second variable;*
(ii) *$T_{2I}(x, y) \le z$ iff $y \le I(x, z)$ for any $x, y, z \in [0, 1]$;*
(iii) *$T_{2I}(x, y) = \min\{t \in [0, 1] \mid y \le I(x, t)\}$ for any $x, y \in [0, 1]$;*
(iv) *$y \le I(x, T_{2I}(x, y))$ for any $x, y \in [0, 1]$.*

It is clear from the above theorem that a right-continuous fuzzy implication I in its second variable and its induced operator T_{2I} satisfy the GMP in the form $T_{2I}(x, I(x, y)) \le y$ for any $x, y \in [0, 1]$.

Theorem 3.5. *Let fuzzy implication I on* $[0, 1]$ *be right-continuous in its second variable; then*

(i) *$T_{1I}(I(y, T_{1I}(x, y)), y) \ge T_{1I}(x, y)$, $I(x, T_{1I}(I(x, y), x)) = I(x, y)$ for any $x, y \in [0, 1]$;*
(ii) *$T_{2I}(x, I(x, T_{2I}(x, y))) \ge T_{2I}(x, y)$, $I(x, T_{2I}(x, I(x, y))) = I(x, y)$ for any $x, y \in [0, 1]$.*

Proof. (i) Since I is right-continuous in its second variable, it is infinitely inf-distributive in this variable. For any $x, y \in [0, 1]$, we obtain $T_{1I}(I(y, T_{1I}(x, y)), y) = T_{1I}(I(y, \inf\{t \in [0, 1]|x \le I(y, t)\}), y) = T_{1I}(\inf\{I(y, t)|t \in [0, 1], x \le I(y, t)\}, y) \ge T_{1I}(x, y)$, since from (3.1) we know that T_{1I} is increasing in its first variable. $I(x, T_{1I}(I(x, y), x)) = I(x, \inf\{t \in [0, 1]|I(x, y) \le I(x, t)\}) = \inf\{I(x, t)|t \in [0, 1], I(x, y) \le I(x, t)\} = I(x, y)$.

(ii) Similar to (i). □

We now give the conditions under which T_{1I} and T_{2I} are left-continuous respectively in the first and in the second variable.

Theorem 3.6. *Assume that fuzzy implication I on* $[0, 1]$ *is right-continuous in its second variable; then*

(i) *T_{1I} is left-continuous in its first variable and $I_{1T_{1I}} = I$;*
(ii) *T_{2I} is left-continuous in its second variable and $I_{2T_{2I}} = I$.*

Proof. (i) First, by (3.1), we know that T_{1I} is increasing in its first variable. It is enough to prove that T_{1I} is infinitely sup-distributive in its first variable. By means of Theorem 3.3(i) and (ii), we have, for any $y \in [0, 1]$ and any subset X of $[0, 1]$,

$$\begin{aligned} T_{1I}(\sup_{x\in X} x, y) &= \inf\left\{t \in [0, 1] \mid \sup_{x\in X} x \le I(y, t)\right\} \\ &= \inf\{t \in [0, 1] \mid \forall x \in X, x \le I(y, t)\} = \inf\{t \in [0, 1] \mid \forall x \in X, T_{1I}(x, y) \le t\} \\ &= \inf\left\{t \in [0, 1] \mid \sup_{x\in X} T_{1I}(x, y) \le t\right\} = \sup_{x\in X} T_{1I}(x, y). \end{aligned}$$

(ii) Similar to (i). □

From Theorems 3.1 and 3.6, we have the following results.

Corollary 3.1. *Assume that fuzzy implication I on* $[0, 1]$ *is right-continuous in its second variable and satisfies* (NP)*; then T_{1I} is a type-A ps-t-norm which is left-continuous in its first variable and T_{2I} is a type-B ps-t-norm which is left-continuous in its second variable.*

Under the conditions of Corollary 3.1, if fuzzy implication I is also left-continuous in its first variable, then it is easy to verify that T_{1I} and T_{2I} are left-continuous in each variable.

In what follows, we propose another method inducing type-A ps-t-norms, type-B ps-t-norms and t-seminorms from fuzzy implications on [0, 1].

Theorem 3.7. *Let I be an implication on* $[0, 1]$ *and the negation* $N_I : [0, 1] \to [0, 1]$ *defined by* $N_I(x) = I(x, 0)$ *for any* $x \in [0, 1]$ *be involutive. We define the mapping* $T : [0, 1]^2 \to [0, 1]$ *as follows:*

$$T(x, y) = N_I(I(x, N_I(y))), \quad x, y \in [0, 1]. \tag{3.3}$$

Then

(i) *T is a type-A ps-t-norm;*
(ii) *if I satisfies* (NP)*, then T is a t-seminorm, and hence a type-B ps-t-norm.*

Proof. First of all, it follows from (3.3) that T is increasing in both variables.

(i) Since N_I is involutive, we have, for any $x \in [0, 1]$, $T(x, 1) = N_I(I(x, N_I(1))) = N_I(I(x, 0)) = x$, $T(x, 0) = N_I(I(x, 1)) = N_I(1) = 0$.

(ii) Since I satisfies (NP) and N_I is involutive, we get, for any $x \in [0, 1]$, $T(1, x) = N_I(I(1, N_I(x))) = N_I(N_I(x)) = x$, $T(0, x) = N_I(I(0, N_I(x))) = N_I(1) = 0$. □

4. Conclusion

In this paper, we have introduced the definitions of two classes of pseudo-t-norms called type-A and type-B ps-t-norms and their respective residual operators. We have proved that these residual operators are fuzzy implications and that they satisfy the left neutral property, and we have given a series of equivalent conditions of left-continuity w.r.t. the first or second variable for these two classes of pseudo-t-norms. Two kinds of induced operators from fuzzy implications have been defined, and the conditions under which they are type-A ps-t-norms, type-B ps-t-norms and t-seminorms have been given. For a fuzzy implication, a series of equivalent conditions of right-continuity w.r.t. its second variable have been established. Another method inducing type-A ps-t-norms, type-B ps-t-norms and t-seminorms by implications has been proposed. In our future study, we will generalize this work to the case of uninorms. We will define pseudo- and semi-uninorms and further define and discuss their residual operators. These works will bring benefit for approximate reasoning, information aggregation and other application areas.

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